NEW RESULTS ON QUALITY MEASURES FOR PLANAR QUADRILATERAL MESHES

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ABSTRACT: This article examines the quality assessment of planar quadrilateral mesh elements in a comprehensive way. First, an analytic characterization of quadrangular shape is provided, and existing concepts of *stretching* and *skewness*, earlier proposed by for specific geometries, are generalized. Then, two triangle quality measures are extended to quadrilaterals and their respective extremal and asymptotic behaviors examined, showing in particular that they cannot detect, if needed, triangular degeneracy of a quadrilateral. An existing quality measure is then discussed, which is able to handle this case. In particular, an unbalanced asymptotic behavior is demonstrated, justifying the need for a new approach. Toward this goal, the triangle quality measure based on FROBENIUS norm is modified in order to replace equilateral reference element by right isosceles triangles, with control on the specific right angle. Finally, two new quadrilateral quality measures are designed and examined using these results. Numerical results illustrate the matter.

KEY WORDS: Quadrilateral, quadrilateral meshing, mesh quality.

Introduction

It is now widely known that the geometric properties of the mesh supporting a finite element or finite volume computation directly impact the accuracy of the numerical result. For example, in the particular case of finite element analysis of elliptic problems, [3] shows that accuracy of the approximate solution is directly related to a geometric estimate of mesh elements, generally referred to as *aspect ratio*, in the case where these elements are simplicial.
Although triangular and tetrahedral quality measures have been extensively discussed, in particular in [1, 2, 5, 6, 8, 10, 11], little work has been done concerning planar quadrilaterals. For such elements, [12] proposes estimating the quality by means of the *aspect ratio*, the *skewness* and the *stretching factor*. Although natural for some particular geometries, such measures are not well defined in a general context. Alternate quality estimates have been introduced by [7], but without providing comparisons.
After recalling a few useful geometrical results, this article firstly discusses the analytical characterization of planar quadrilaterals. In particular, asymptotic

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degeneracy cases are examined. The modification of triangle quality measures based on matrix norms, proposed in [9] in order to adapt these measures to the case where reference elements are right isosceles triangles, is provided *in extenso*. This adaptation is motivated by the idea of extending them to quadrilateral quality measure. Then, the first quadrangle shape estimates to be considered are skewness and stretching factor. They are examined in detail and it is explained why they cannot provide quadrangle quality measures in a general context. Because of these limitations, extensions of some triangle quality measures to quadrilaterals are then provided. These quality measures are fully analyzed and it is shown that they satisfy the desired extremal and, depending on the context, asymptotic properties. When these asymptotic properties do not comply with the requirements of the application, an alternate estimate introduced in [7] is examined in detail. Finally, the quadrilateral quality measures based on the FROBENIUS matrix norm, previously tailored to right isosceles reference elements, are defined and studied. Several numerical examples illustrate the theoretical results.

1 Preliminaries

1.1 Triangles

In all that follows, t denotes a triangle with vertices E, F and G, area $\mathcal{A}_t$, semiperimeter p_t, inradius r, circumradius R, edge lengths $e = FG$, $f = EG$ and $g = EF$, and we denote the angle at vertex E (resp. F, G) as η (resp. ϕ, ψ) and the radius of the inscribed (resp. circumscribed) circle of t as r (resp. R). In addition, the vertices E, F and G have respective coordinates (x_E, y_E), (x_F, y_F) and (x_G, y_G) in an arbitrary orthonormal affine reference frame parallel to the plane of the triangle t. The following standard norm-like notations will also be used:

$$\begin{array}{rcl} |t|_0 & = & \min(e,f,g) \\ |t|_2 & = & \sqrt{e^2+f^2+g^2} \\ |t|_\infty & = & \max(e,f,g). \end{array} \tag{1.1}$$

Some results from elementary geometry are assumed without proof (see for example [4] for proofs and details). In particular, the following well-known relations will be used:

$$2R = \frac{efg}{2\mathcal{A}_t} = \frac{e}{\sin\eta} = \frac{f}{\sin\phi} = \frac{g}{\sin\psi}, \tag{1.2}$$

where $\mathcal{A}_t$ is given by

$$\mathcal{A}_t = rp_t, \tag{1.3}$$

as well as by HERON's formula:

$$\mathcal{A}_t = \sqrt{p_t(p_t-e)(p_t-f)(p_t-g)}. \tag{1.4}$$

Finally, it is recalled that the edge ratio is defined as:

$$\tau = \frac{|t|_\infty}{|t|_0}, \tag{1.5}$$

see [10] for a study of the behavior of τ, with respect to the extremal angles of t.

1.2 Quadrilaterals

For the sake of conciseness, "quadrilateral" will hereafter mean a "planar, non-degenerate and convex quadrilateral". Such a quadrilateral shall be denoted generically as $q = ABCD$, with area $\mathcal{A}$, semiperimeter p, edges of lengths $a = AB$, $b = BC$, $c = CD$ and $d = DA$ and denote the angle at vertex A (resp. B, C, D) as α (resp. β, γ, δ) and θ the arithmetic mean of either α and γ, or β and δ, called the *torsion*. Norm-like notations, similar to those for triangles, will also be used:

$$\begin{array}{rcl} |q|_0 & = & \min(a,b,c,d) \\ |q|_2 & = & \sqrt{a^2+b^2+c^2+d^2} \\ |q|_\infty & = & \max(a,b,c,d). \end{array} \tag{1.6}$$

Remark 1.1. The computation of $\mathcal{A}$ can be performed in several different ways. In particular, it seems natural to decompose q in two triangles, whose respective areas are obtained in a straightforward manner obtained. Although twice more costly, it is not a bad idea to compute the four possible triangular areas, since this allows to detect, on the fly, whether or not q is convex, non-convex, skew, degenerate (*cf.* [7] for details). Practically speaking, this is of the greatest interest, since both topological consistency checking and geometrical quality measurement can be done at the same time.

Most of the useful metric equalities of triangles do not extend to quadrilaterals, and this is the first obstacle to the generalization of results such as those presented in [10] in the case of triangles. Nevertheless, HERON's formula can be generalized for quadrilaterals:

$$\mathcal{A} = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}. \tag{1.7}$$

Remark 1.2. This gives an opportunity to discuss the choice of α and γ in the definition of θ. It is well known that the sum of the four angles of a convex quadrilateral is equal to 2π; hence, $\theta_1 = \frac{\alpha+\gamma}{2}$ and $\theta_2 = \frac{\beta+\delta}{2}$ are supplementary. It follows in particular that θ_1 and θ_2 have opposite cosines, thus equal squared cosines. Therefore, whatever pair of opposite angles is picked in order to define θ, (1.7) returns the same result, *i.e.*, it is symmetrical.

1.3 Cocyclicity

It might be useful to recall some results about planar cocyclicity, and in particular the fact that, unlike triangles, quadrilaterals cannot necessarily be inscribed in a circle. More precisely,

Definition 1.1. A quadrilateral is *cyclic* if it can be inscribed in a circle.

Example 1.1. The quadrilateral $\textcircled{q}$, illustrated Figure 1, which is homothetic to the quadrilateral with successive vertices coordinates $(\sqrt{3}, 1)$, $(\sqrt{2}, \sqrt{2})$, $(1, \sqrt{3})$ and $(-\sqrt{3}, 1)$, is cyclic.

A demonstration of the beautiful *Inscribed Angle Theorem* can be found in any good elementary geometry handbook; using the classical notation for vector angles, it comes as follows:

Theorem 1.1. *P, Q, R and S are cocyclic if and only if:*

$$(\overrightarrow{PQ}, \overrightarrow{PS}) \equiv (\overrightarrow{RQ}, \overrightarrow{RS}) \quad [\pi] \tag{1.8}$$

and allows to readily deduce the

Corollary 1.2. *q is cyclic if and only if α and γ are supplementary.*

Remark 1.3. According to Remark 1.2, Corollary 1.2 can be equivalently expressed using β and δ instead of α and γ.

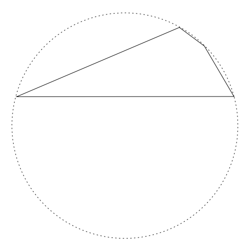

Figure 1: Cyclic quadrilateral $\textcircled{q}$.

1.4 Deriving quadrilateral quality from triangle quality

A natural approach to mesure the quality of any given non-degenerate convex quadrilateral consists in seeing it as a pair of two non-degenerate triangles sharing one common edge, which is also a diagonal of the quadrilateral. Hence, an apparently good idea would be to examine the qualities of these two triangles, but which quality ? Generally speaking (see [10] for a notable exception), the quality of a triangle is considered to be optimal[1] only for equilateral triangles. Unfortunately, the following example shows that using such triangle quality measurements for quadrilaterals is not straightforward.

Example 1.2. Figure 2 illustrates the case where q is a rhombus, such that its shortest diagonal has the same length as its edges, hence denoted as $\Diamond$. By

[1] more precisely, reaches its strict and unique *minimum*, 1.

definition, $\diamondsuit$ can be decomposed either into two equilateral triangles or into two obtuse isosceles triangles. In the general sense of triangle quality, the former case is considered as optimal, while the latter is far from this. In other words, the choice of the particular partition of q in two triangles has an effect over the resulting quadrilateral quality measurement; which one shall be chosen ?

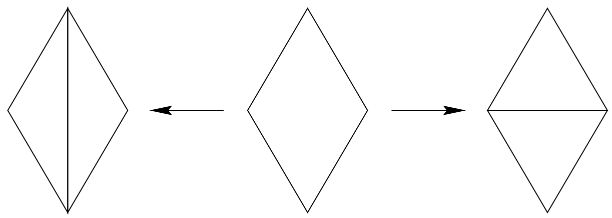

Figure 2: The two possible triangulations of the same rhombus $\diamondsuit$.

The only certainty at this point is that either both triangular decompositions of q must be taken into account, or another approach of quadrilateral quality, independent from the underlying triangles, must be used.

2 The quadrilateral space

The aim of this section is to define a proper set on which the analysis of quadrilateral quality measures will be relevant. In particular, the concept of equivalence class will be used.

2.1 Homothecy equivalence classes

A quadrilateral can be seen as either a geometric or an analytic object. Although the former is certainly more intuitive, the latter allows the use of calculus to perform an analysis of quadrilateral quality measures. A consistent analytic representation of quadrilaterals is one that is bijective with the set of geometric quadrilaterals. In addition, quality measures in the general sense (*cf.* [7, 10]) do not depend on size, but only on shape; in other words, they are invariant through homothecy.
Hence, it is more suitable to use an analytical representation of quadrilaterals, up to homothecy.
Equality up to homothecy is an equivalence class, in the strict mathematical sense: reflexive, symmetrical and transitive. Therefore, it allows the definition of equivalence classes of quadrilaterals; in particular, any quadrilateral q belongs to one and only one equivalence class, and this class is the set of all quadrilaterals which are homothetic to q. In addition, any equivalence class can be represented by one of its elements, *e.g.*, the only quadrilateral with unitary semiperimeter,

as illustrated by Figure 3. In other words, considering only the set of such quadrilaterals is sufficient for quality measure analysis, since these measures must be invariant to scaling. This set, which is in fact the set of all equivalence classes[2], is simply denoted as Q_1.

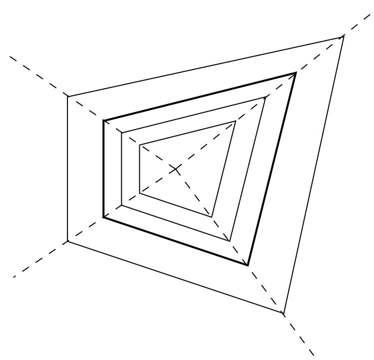

Figure 3: Equality up to homothecy: the unique quadrilateral with unitary semiperimeter represents the entire homothecy class.

2.2 Analytic characterization

Since any edge length of a non-degenerate quadrilateral q is strictly smaller than the sum of the three other ones, it follows that

$$\frac{p}{a} = \frac{a+b+c+d}{2a} = \frac{1}{2} + \frac{b+c+d}{2a} > \frac{1}{2} + \frac{a}{2a} = 1 \tag{2.1}$$

and, for the same reasons, $\frac{p}{b} > 1$, $\frac{p}{c} > 1$ and $\frac{p}{d} > 1$. Hence, denoting as x, y and z the ratios between three edge lengths to the semiperimeter of q, *e.g.*, $x = \frac{a}{p}$, $y = \frac{b}{p}$ and $z = \frac{c}{p}$, it is clear that $x < 1$, $y < 1$ and $z < 1$. In addition,

$$x + y + z = \frac{a+b+c}{p} = \frac{2p-d}{p} = 2 - \frac{d}{p}, \tag{2.2}$$

whence

$$1 < x + y + z < 2 \tag{2.3}$$

and

$$0 < 2 - x - y - z = \frac{d}{p} < 1. \tag{2.4}$$

Hence, on the one hand, the quadrilateral q_1 with consecutive edge lengths x, y, z and $2 - x - y - z$ is homothetic to q (with ratio p); on the other hand, its

[2]or, equivalently, the quotient space of the set of quadrilaterals by the equality up to homothecy.

semiperimeter is obviously unitary thus $q_1 \in \mathsf{Q}_1$. In other words, q_1 is the class representative of q, on which quality measure analysis shall be performed.
Now, the knowledge of the four edge lengths of a quadrilateral is not sufficient to determine its shape[3]. For example, knowing that a quadrilateral has four equal edge lengths only allows to conclude that it is a rhombus; nothing is know about the angles of this rhombus which might be, in particular, a square.

Remark 2.1. This makes a noticeable difference with the case of triangles, for which there is a bijection between, on the one hand, the ratios between edge lengths and, on the other hand, the angles of this triangle.

In fact, the knowledge of a, b, c, d and, *e.g.*, the angle α, completely determines q and, in particular, the other angles β, γ and δ, for an obvious reason: knowing α allows to determine one of the diagonals, therefore the triangle opposite to α is fully determined by its three edges, according to Remark 2.1. Simply stated, this means that, in addition to edge lengths, quadrilaterals have one and only one other degree of freedom, provided by any of their four angles. Thus, the set of quadrilaterals can be seen as a subset of $\mathbb{C}\ \mathbb{R}^5$ and, since angles are invariant through homothecy, Q_1 as a subset of $\mathbb{R}^4$. In other words, the quadrilateral shape space is four-dimensional[4].

2.3 Asymptotic configurations

As Q_1 is four-dimensional, it is not as natural to examine the asymptotic behavior, as in the case of triangles, in particular because it is impossible to visualize[5] an hypersurface of R^5. Nevertheless, examining some usual configurations can provide useful informations concerning some lower-level shape estimates, such as the torsion or the stretching factor. For example, the set of parallelograms up to homothecy has the nice property of being a 2-dimensional subset of Q_1, since $a = c$ and $b = d$. Moreover, these two degrees of freedom can be straightforwardly seen as the stretching factor and the torsion, and this allows a direct relationship between the behavior of a quality measure to these parameters.
Another interesting asymptotic case occurs when two consecutive edges of q tend towards alignment, in other words when q gets close to being a triangle t, as shown Figure 4. Although this is a somewhat intuitive notion, this situation can be formalized as follows: precisely,

Definition 2.1. A *triangular degeneracy* occurs when one, and only one, angle of a quadrilateral is close to π and none is close to 0.

Triangular degeneracy might be considered as either acceptable or not, *i.e.*, the quality measure should either remain bounded or diverge to $+\infty$, respectively. For example, in the context of hybrid meshes, it could be useful to have a triangle-quadrilateral quality measure, which would not diverge in the case of

[3] or, equivalently, the knowledge of x, y and z is not sufficient to determine the homothecy equivalence class.

[4] intuitively, since no vector field structure has been properly defined.

[5] at least, for human beings.

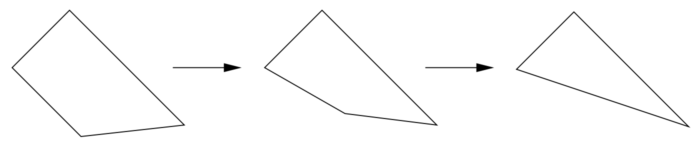

Figure 4: Triangular degeneracy.

triangular degeneracy of q, but rather continuously match along the transition between the two kinds of polygons. To the contrary, most quadrilateral finite elements solvers would handle triangularily degenerated elements poorly.
Provided these results concerning the analytical characterization of quadrilateral shape, it is now possible to examine precisely the extremal and asymptotic properties of several possible quadrilateral quality measures.

3 Edge ratio

It seems natural to extend τ, the edge ratio which has been defined for triangles, to quadrilaterals. In this case, we have:

$$\tau = \frac{|q|_\infty}{|q|_0}. \tag{3.1}$$

Remark 3.1. τ can be seen as a generalization of the stretching factor, defined in [12] for specific geometries. Therefore, it seems rather intuitive to extend the use of this expression to all quadrilaterals, and τ will be also referred to as the *stretching factor* of q.

3.1 *Extremum*

By definition, $\tau \geq 1$, with equality if and only if $|q|_\infty = |q|_0$, *i.e.*, $\min(a,b,c,d) = \max(a,b,c,d)$, thus $a = b = c = d$. Therefore, τ has a unique absolute *minimum*, 1, attained only by rhombii. Squares are only a particular minimal case.

3.2 Asymptotic behavior

3.2.1 Parallelogram

Obviously, if q is a parallelogram, then either $\tau = \frac{b}{a}$ or $\tau = \frac{a}{b}$. In particular, τ is completely independent of the angles of q, which is not the case when considering the edge ratio of a triangle. Intuitively, the behavior of τ can be seen as one-dimensional, with respect to the four dimensions of Q_1. This is why it has been proposed for specific geometries, in particular for rectangles: in terms of homothecy equivalence classes, they are completely characterized by their length to width ratio, *i.e.*, by τ. Minimal for and only for squares, it increases and tends to $+\infty$ as the rectangle stretches, hence its name. To the contrary, τ is unable to provide any valuable when q is a rhombus: either it is square or very flattened, with θ as close to 0 as desired, it is considered as optimal by τ.

3.2.2 Triangle degeneracy

If q degenerates towards a triangle t, the two following different cases may occur:

i. if two vertices tend to merge, then $|q|_0$ tends to 0. Thereore, τ diverges to $+\infty$;

ii. otherwise, if the triangle degeneracy occurs with no edge lengths tending to 0, then $|q|_0$ tends to a non-zero value while $|q|_\infty$ remains bounded, preventing τ from degenerating.

In addition, a continuation of τ to the edge ratio of the triangle occurs only if $|q|_0$ and $|q|_\infty$ remain unchanged, when jumping from q to t.
Clearly, neither the asymptotic nor extremal properties of q correspond to what is expected from a quadrilateral quality measure. However, it is a very intuitive dimension of the quadrilateral space, thus useful for examining the behavior of such measures.

Remark 3.2. The edge ratio also provides a triangle quality measure, optimal only for equilateral triangles. It has the original property of accepting flattened elements, but not needles for which it diverges. See [10] for details.

4 Skewness

Skewness does not seem to have been properly defined in a general context. We propose the following definition:

$$\sigma = \frac{1}{1 - \left|1 - \frac{2\theta}{\pi}\right|}. \tag{4.1}$$

As mentioned in Remark 1.2, the two possible values of θ, θ_1 and θ_2, are supplementary thus

$$\left|1 - \frac{2\theta_1}{\pi}\right| = \left|1 - \frac{2(\pi - \theta_2)}{\pi}\right| = \left|-1 + \frac{2\theta_2}{\pi}\right| = \left|1 - \frac{2\theta_2}{\pi}\right|, \tag{4.2}$$

from which it follows that σ is independent from the particular choice of either θ_1 or θ_2 for θ. More precisely, σ is symmetrical with respect to $\frac{\pi}{2}$.

4.1 *Extremum*

By definition, one has

$$
\begin{aligned}
0<\theta<\pi \quad &\Longleftrightarrow \quad 0<\frac{2\theta}{\pi}<2 && (4.3)\\
&\Longleftrightarrow \quad -1<1-\frac{2\theta}{\pi}<1 && (4.4)\\
&\Longleftrightarrow \quad 0\leq\left|1-\frac{2\theta}{\pi}\right|<1 && (4.5)\\
&\Longleftrightarrow \quad 0<1-\left|1-\frac{2\theta}{\pi}\right|\leq 1 && (4.6)\\
&\Longleftrightarrow \quad \sigma\geq 1. && (4.7)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\sigma=1 \quad &\Longleftrightarrow \quad 1-\left|1-\frac{2\theta}{\pi}\right|=1 && (4.8)\\
&\Longleftrightarrow \quad \left|1-\frac{2\theta}{\pi}\right|=0 && (4.9)\\
&\Longleftrightarrow \quad \theta=\frac{\pi}{2}, && (4.10)
\end{aligned}
$$

thus σ reaches its unique and strict *minimum* when $\theta=\frac{\pi}{2}$; in other words, using Corollary 1.2, for cyclic quadrilaterals.
Although squares and, more generally, rectangles, are cyclic, the converse is obviously untrue. In particular, the quadrilateral @, introduced in Example 1.1 and illustrated Figure 1, is considered as optimal by σ, and could even be flattened as much as desired, provided its four vertices remain cocyclic.

4.2 Asymptotic behavior

It is clear that σ strictly decreases (resp. increases) on $]0,\frac{\pi}{2}]$ (resp. $[\frac{\pi}{2},\pi[$); moreover,

$$\lim_{\theta\to 0^+}\frac{1}{1-\left|1-\frac{2\theta}{\pi}\right|}=\lim_{\theta\to\pi^-}\frac{1}{1-\left|1-\frac{2\theta}{\pi}\right|}=+\infty, \tag{4.11}$$

since $\sigma(\theta)=\frac{\pi}{2\theta}$ $\left(\text{resp. } \frac{\pi}{2(\pi-\theta)}\right)$ when $\theta\to 0^+$ (resp. $\theta\to\ \pi^-$).
Figure 5 represents the graph of skewness *vs.* torsion. The previously mentioned symmetry of σ results in a symmetric graph, and the fact that the absolute value is not $\mathcal{C}^1$ in 0 results in a salient point[6] when $\theta=\frac{\pi}{2}$.
Rectangles are only a particular case of inscribed quadrilaterals; therefore, any rectangle is optimal, however stretched it is. Conversely, any non-rectangular parallelogram, *e.g.*, any non-square rhombus, is not optimal for σ. In fact, σ can be also seen as one-dimensional, but along a different direction of Q_1 than τ.

[6] *i.e.*, left and right tangents at that point are not aligned.

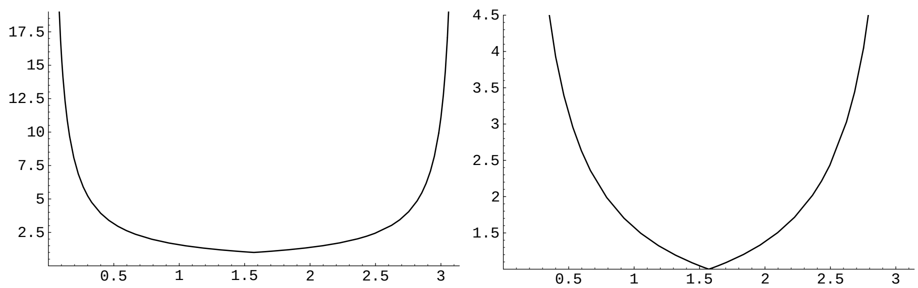

Figure 5: Graphs at two different scales of σ as a function of θ.

In particular, triangular degeneracy is not caught by σ: if, say, $\alpha \to \pi$ then, according to Definition 2.1, γ and, consequently $\theta_1 = \frac{\alpha+\gamma}{2}$, tend neither to 0 nor π; hence, σ does not diverge.

5 Edge to inradius

Among triangle qualities that extend naturally to quadraliterals is the comparison of edge lengths with inradius. Of course, any convex quadrilateral does not have, in general, an inscribed circle[7] and, hence, it might seem paradoxical to attempt to extend such edge to inradius comparisons to quadrilaterals. However, in the case of a non-degenerate triangle t, it follows from (1.3) that:

$$\frac{p_t}{r} = \frac{p_t^2}{\mathcal{A}_t}. \tag{5.1}$$

Therefore, this quality measure can be directly extended to q:

$$\zeta = \frac{p^2}{\mathcal{A}}. \tag{5.2}$$

Similarily, the *aspect-ratio* can be extended to q:

$$\iota = \frac{p|q|_\infty}{\mathcal{A}}, \tag{5.3}$$

and these two measures are related *via* the following inequality:

$$\zeta = \frac{p(a+b+c+d)}{2\mathcal{A}} \leq \frac{4p|q|_\infty}{2\mathcal{A}} = 2\,\iota \tag{5.4}$$

with equality if and only if $p = 2|q|_\infty$, *i.e.* if and only if q is a rhombus.

[7] In fact, such an incircle exists if and only if $\alpha + \gamma = \beta + \delta$.

5.1 *Extremum*

Combining (1.7) and (1.3) gives:

$$\zeta = \sqrt{\frac{p^4}{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}} \tag{5.5}$$

and, as for triangles, it is much more convenient[8] to try to minimize $h = \frac{1}{\zeta^2}$ rather than to maximize ζ. Clearly, h is a function of only the five variables a, b, c, p and θ, and can be expressed as follows:

$$h(a,b,c,p,\theta) = \frac{(p-a)(p-b)(p-c)(a+b+c-p) - abc(2p-a-b-c)\cos^2\theta}{p^4} \tag{5.6}$$

and it is clear that, for any $a \in \mathrm{IR}_+^*$, $h(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}, 1, \theta) = h(a,b,c,p,\theta)$. This means that h only depends on four variables, given by the ratios of three edges lengths to the semiperimeter, plus the torsion angle θ. In other words, h is invariant through homothecy, as expected, thus ζ is non-dimensional. We therefore examine the variations of $\tilde{h} : \mathsf{Q}_1 \to \mathrm{IR}$, where

$$\tilde{h}(x,y,z,\theta) = (1-x)(1-y)(1-z)(x+y+z-1) + xyz(x+y+z-2)\cos^2\theta \tag{5.7}$$

whose first order derivatives are:

$$\frac{\partial \tilde{h}}{\partial x}(x,y,z,\theta) = (2x+y+z-2)\big((1-y)(1-z) + yz\cos^2\theta\big) \tag{5.8}$$

$$\frac{\partial \tilde{h}}{\partial y}(x,y,z,\theta) = (x+2y+z-2)\big((1-x)(1-z) + xz\cos^2\theta\big) \tag{5.9}$$

$$\frac{\partial \tilde{h}}{\partial z}(x,y,z,\theta) = (x+y+2z-2)\big((1-x)(1-y) + xy\cos^2\theta\big) \tag{5.10}$$

$$\frac{\partial \tilde{h}}{\partial \theta}(x,y,z,\theta) = xyz(x+y+z-2)\sin 2t. \tag{5.11}$$

According to (2.3), $x+y+z < 2$, hence, since none of x, y nor z are null, the stationary-point condition implies that $2\theta \in \pi\mathbb{Z}$ thus, since $2\theta \in]0,\pi[$, necessarily $\theta = \frac{\pi}{2}$. Hence, $(x,y,z,\frac{\pi}{2})$ is a stationary-point if and only if

$$\left\{\begin{array}{rcl} (2x+y+z-2)(1-y)(1-z) & = & 0 \\ (x+2y+z-2)(1-x)(1-z) & = & 0 \\ (x+y+2z-2)(1-x)(1-y) & = & 0 \end{array}\right. \tag{5.12}$$

which is equivalent, since neither x nor y nor z is equal to 1, to

$$\left\{\begin{array}{rcl} 2x+y+z & = & 2 \\ x+2y+z & = & 2 \\ x+y+2z & = & 2 \end{array}\right. \tag{5.13}$$

[8] and, obviously, equivalent.

whose only solution is, clearly, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Therefore, $\tilde{h}$ has a unique stationary point, when the quadrilateral is a square. In order to check whether this case corresponds, as expected, to a *minimum*, one has to make sure that the hessian matrix is positive definite. The second order derivatives of $\tilde{h}$ are given by:

$$\frac{\partial^2 \tilde{h}}{\partial x^2}(x, y, z, \theta) = 2(y + z - yz - 1 + yz\cos^2\theta) \tag{5.14}$$

$$\frac{\partial^2 \tilde{h}}{\partial y^2}(x, y, z, \theta) = 2(x + z - xz - 1 + xz\cos^2\theta) \tag{5.15}$$

$$\frac{\partial^2 \tilde{h}}{\partial z^2}(x, y, z, \theta) = 2(x + y - xy - 1 + yz\cos^2\theta) \tag{5.16}$$

$$\frac{\partial^2 \tilde{h}}{\partial \theta^2}(x, y, z, \theta) = 2xyz(2 - x - y - z)\cos 2t \tag{5.17}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial y}(x, y, z, \theta) = (1-z)(2x + 2y + z - 3) + z(2x + 2y + z - 2)\cos^2\theta \tag{5.18}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial z}(x, y, z, \theta) = (1-y)(2x + y + 2z - 3) + y(2x + y + 2z - 2)\cos^2\theta \tag{5.19}$$

$$\frac{\partial^2 \tilde{h}}{\partial y \partial z}(x, y, z, \theta) = (1-x)(x + 2y + 2z - 3) + x(x + 2y + 2z - 2)\cos^2\theta \tag{5.20}$$

$$\frac{\partial^2 \tilde{h}}{\partial x \partial \theta}(x, y, z, \theta) = yz(2 - 2x - y - z)\sin 2t \tag{5.21}$$

$$\frac{\partial^2 \tilde{h}}{\partial y \partial \theta}(x, y, z, \theta) = xz(2 - x - 2y - z)\sin 2t \tag{5.22}$$

$$\frac{\partial^2 \tilde{h}}{\partial z \partial \theta}(x, y, z, \theta) = xy(2 - x - y - 2z)\sin 2t \tag{5.23}$$

and, therefore, the hessian matrix of $\tilde{h}$ in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})$ is:

$$H_{\tilde{h}_\square} = -\frac{1}{8}\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.24}$$

and is clearly negative-definite; more precisely, the characteristic polynomial of the matrix

$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 4 & 2 & 0 \\ 2 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.25}$$

is $(X - 1)(X - 2)^2(X - 8)$, thus the eigenvalues of $H_{\tilde{h}_\square}$ are $-\frac{1}{4}$ (double), $-\frac{1}{8}$ and -1. Hence, the stationary point is a *maximum*, meaning that ζ reaches its only *minimum* for squares; in addition, this *minimum*, denoted as $\zeta_\square$, is strict. From this extremal property of ζ can be deduced another one for ι: firstly, since (5.4) is an equality only for rhombii, it is the case for squares, thus $\zeta_\square = 2\iota_\square$.

Now, combining the minimization of ζ with (5.4), leads to:

$$\iota \geq \frac{\zeta}{2} \geq \frac{\zeta_\square}{2} = \iota_\square, \tag{5.26}$$

showing that, as ζ, ι is minimal for, and only for, squares.

5.2 Asymptotic behavior

5.2.1 Parallelogram

In this case, $\alpha = \gamma$ and $\beta = \delta$ thus $\theta = \alpha$ or $\theta = \beta$. Moreover, α and β are supplementary, hence have the same sine. Therefore, the area of q is given by:

$$\mathcal{A} = ab \sin\theta = |q|_0 |q|_\infty \sin\theta, \tag{5.27}$$

thus ζ can be expressed as:

$$\zeta = \frac{(|q|_0 + |q|_\infty)^2}{|q|_0|q|_\infty \sin\theta} = \frac{1}{\sin\theta}\left(\tau + \frac{1}{\tau} + 2\right) \tag{5.28}$$

and ι as:

$$\iota = \frac{(|q|_0 + |q|_\infty)|q|_\infty}{|q|_0|q|_\infty \sin\theta} = \frac{1}{\sin\theta}\left(\tau + 1\right). \tag{5.29}$$

5.2.2 Rectangle

If q is a rectangle, then (5.28) and (5.29) respectively become $\zeta = \tau + \frac{1}{\tau} + 2$ and $\iota = \tau + 1$. Therefore, ζ and ι are two functions of τ, strictly increasing over $[1, +\infty[$, respectively denoted as $\zeta_\square$ and $\iota_\square$. Moreover,

$$\zeta_\square(\tau) \underset{\tau\to+\infty}{\sim} \tau \underset{\tau\to+\infty}{\sim} \iota_\square(\tau), \tag{5.30}$$

which means that they asymptotically behave as τ for rectangles. In particular, $\zeta_\square$ and $\iota_\square$ diverge to $+\infty$ when τ does.

5.2.3 Rhombus

If q is a rhombus, then it follows from (5.28) that:

$$\zeta = 2\iota = \frac{4}{\sin\theta} \tag{5.31}$$

which gives a direct relation between these quality measures and the torsion of the rhombus, since they reduce to two functions $\zeta_\diamond$ and $\iota_\diamond$, depending only on θ, with $\zeta_\diamond = 2\iota_\diamond$. Obviously, they both strictly increase (resp. decrease) on $]0, \frac{\pi}{2}]$ (resp. $[\frac{\pi}{2}, \pi[$). They also diverge to $+\infty$ at the bounds of $]0, \pi[$ since, using the fact that $\sin\theta = \sin(\pi - \theta)$,

$$\zeta_\diamond(\theta) = 2\iota_\diamond(\theta) \underset{\tau\to0^+}{\sim} \frac{4}{\theta} \tag{5.32}$$

$$\zeta_\diamond(\theta) = 2\iota_\diamond(\theta) \underset{\tau\to\pi^-}{\sim} \frac{4}{\pi - \theta}. \tag{5.33}$$

The relative sensitivities of ζ to stretching and torsion are now discussed, by solving $\zeta_{\square}(\tau) = \zeta_{\diamond}(\theta)$, *i.e*,

$$\tau + \frac{1}{\tau} + 2 = \frac{4}{\sin\theta} \qquad \Longleftrightarrow \qquad \frac{\tau^2 \sin\theta + 2(\sin\theta - 2)\tau + \sin\theta}{\tau \sin\theta} = 0, \tag{5.34}$$

which can be solved either for τ or θ. In the former case, (5.34) reduces to finding the real roots of the polynomial

$$P_\zeta = X^2 \sin\theta + 2(\sin\theta - 2)X + \sin\theta \tag{5.35}$$

whose determinant is

$$\begin{aligned} \Delta_{P_\zeta} &= 4\left((\sin\theta - 2)^2 - \sin^2\theta\right) && (5.36)\\ &= 16(1 - \sin\theta) && (5.37)\\ &\geq 0 && (5.38) \end{aligned}$$

with equality if and only if $\theta = \frac{\pi}{2}$. In this case, 1 is the only root of P_ζ, *i.e.*, q is a square. Otherwise, $0 < 1 - \sin\theta < 1$, hence P_ζ has two distinct real roots; in addition, $1 - \sin\theta < \sqrt{1 - \sin\theta}$, thus

$$2 - \sin\theta - 2\sqrt{1 - \sin\theta} < \sin\theta < 2 - \sin\theta + 2\sqrt{1 - \sin\theta}. \tag{5.39}$$

Combining (5.39) with the fact that, by definition, $\tau \geq 1$, it follows that, for any given $\theta \in \,]0, \pi[$, (8.18) has the following unique solution:

$$\tau = \frac{2 - \sin\theta + 2\sqrt{1 - \sin\theta}}{\sin\theta} = \frac{(\sqrt{1 - \sin\theta} + 1)^2}{\sin\theta}. \tag{5.40}$$

Moreover, solving (5.34) for θ obviously brings two possible solutions, $\arcsin \frac{4\tau}{(\tau+1)^2}$ and $\pi - \arcsin \frac{4\tau}{(\tau+1)^2}$. In fact, they both correspond to the same shape for q, since they are supplementary.

5.2.4 Triangular degeneracy

In the case where q degenerates towards a triangle q, it is obvious that both perimeter and area of q tend with their respective counterparts in t. Hence, $\zeta(q)$, in the sense of quadrilateral quality, tends towards $\zeta(t)$, in the sense of triangle quality. In particular, ζ does not diverge during triangular degeneracy. From both extremal properties and asymptotic behavior that have been demonstrated, one can conclude that ζ can be extended as a generic quality measure, suitable for both triangles and quadrilaterals; in addition, a continuous transition between these two kinds of elements is ensured. However, this particular property might be, in some respects, a major drawback since, for some applications, one might wish to avoid triangular degeneracy of quadrilaterals. In this case, the quality measure should diverge to infinity, rather than tend towards the quality of the limiting triangular element.

6 Remembering diagonals

The following quadrilateral quality measure is proposed by [7]:

$$\mathcal{Q} = \frac{|q|_2\, h_{\max}}{\min_i \mathcal{A}_i} \tag{6.1}$$

where $\mathcal{A}_i$ denotes the area of the triangle whose edges are those of q adjacent to vertex i and $h_{\max} = \max\left(AC, BD, |q|_\infty\right)$. In particular, $|q|_\infty \leq h_{\max}$.

6.1 *Extremum*

Applying Cauchy-Schwarz inequality, which is

$$\left(\forall\, (u_1, ..., u_n) \in \mathbb{R}_+^n\right) \quad \sum_{k=1}^{k=n} u_k \leq \sqrt{n \sum_{k=1}^{k=n} u_k^2}, \tag{6.2}$$

to the edge lengths of q shows that

$$a + b + c + d \leq \sqrt{4\left(a^2 + b^2 + c^2 + d^2\right)} \tag{6.3}$$

thus $p \leq |q|_2$, with equality if and only if q is a rhombus. In addition,

$$\min_i \mathcal{A}_i \leq \frac{\mathcal{A}}{2} \tag{6.4}$$

with equality if and only if q is a parallelogram. Hence,

$$\frac{p}{\mathcal{A}} \leq \frac{|q|_2}{2 \min_i \mathcal{A}_i} \tag{6.5}$$

with equality if and only if q is a rhombus. Therefore, if q is not a rhombus,

$$\iota = \frac{p|q|_\infty}{\mathcal{A}} < \frac{|q|_2|q|_\infty}{2 \min_i \mathcal{A}_i} \leq \frac{|q|_2\, h_{\max}}{2 \min_i \mathcal{A}_i} = \frac{\mathcal{Q}}{2}. \tag{6.6}$$

$$\frac{\zeta}{2} \leq \iota = \frac{p|q|_\infty}{\mathcal{A}} \leq \frac{|q|_2\, h_{\max}}{2 \min_i \mathcal{A}_i} = \frac{\mathcal{Q}}{2}. \tag{6.7}$$

and, in particular, $\zeta \leq \mathcal{Q}$.

6.2 Asymptotic behavior

6.2.1 Parallelogram

If q is a parallelogram, then

$$\left(\forall\, i \in \{A, B, C, D\}\right) \quad \min_i \mathcal{A}_i = \frac{\mathcal{A}}{2} = \frac{|q|_0|q|_\infty \sin\theta}{2} \tag{6.8}$$

and, using Al Kashi's Theorem, it is obvious that

$$BD = \sqrt{a^2 + b^2 - 2ab\cos\alpha} \tag{6.9}$$

$$AC = \sqrt{a^2 + b^2 - 2ab\cos\beta} = \sqrt{a^2 + b^2 + 2ab\cos\alpha} \tag{6.10}$$

since α and β are supplementary. Hence,

$$\max(AC, BD) = \sqrt{a^2 + b^2 + 2ab\,|\cos\alpha|} > \max(a, b) = |q|_\infty \tag{6.11}$$

thus

$$h_{\max} = \sqrt{a^2 + b^2 + 2ab\,|\cos\alpha|} = \sqrt{|q|_0^2 + |q|_\infty^2 + 2|q|_0|q|_\infty|\cos\theta|}. \tag{6.12}$$

Finally,

$$|q|_2 = \sqrt{2(a^2 + b^2)} = \sqrt{2(|q|_0^2 + |q|_\infty^2)} \tag{6.13}$$

therefore

$$\mathcal{Q} = \frac{2\sqrt{2(|q|_0^2 + |q|_\infty^2)(|q|_0^2 + |q|_\infty^2 + 2|q|_0|q|_\infty|\cos\theta|)}}{|q|_0|q|_\infty \sin\theta} \tag{6.14}$$

$$= \frac{2\sqrt{2(1+\tau^2)(1+\tau^2+2\tau|\cos\theta|)}}{\tau\sin\theta}. \tag{6.15}$$

6.2.2 Rectangle

In this case, (6.15) shows that $\mathcal{Q}$ becomes the following function of the stretching factor:

$$\mathcal{Q}_\square(\tau) = \frac{2\sqrt{2(1+\tau^2)(1+\tau^2)}}{\tau} = 2\sqrt{2}\left(\tau + \frac{1}{\tau}\right). \tag{6.16}$$

Clearly, $\mathcal{Q}_\square$ strictly increases over $[1, +\infty[$ and

$$\mathcal{Q}_\square(\tau) \underset{\tau\to+\infty}{\sim} 2\sqrt{2}\tau \tag{6.17}$$

hence, in particular, $\lim_{\tau\to+\infty} \mathcal{Q}_\square(\tau) = +\infty$.

6.2.3 Rhombus

In the case where q is a rhombus, then it follows from (6.15) that $\mathcal{Q}$ simplifies as a function of the sole torsion:

$$\mathcal{Q}_\diamond(\theta) = \frac{4\sqrt{2(1+|\cos\theta|)}}{\sin\theta} \tag{6.18}$$

which diverges to $+\infty$ in both 0^+ or π^-. More precisely,

$$\mathcal{Q}_\diamond(\theta) \underset{\tau\to0^+}{\sim} \frac{8}{\theta} \tag{6.19}$$

$$\mathcal{Q}_\diamond(\theta) \underset{\tau\to\pi^-}{\sim} \frac{8}{\pi-\theta}. \tag{6.20}$$

6.2.4 Triangular degeneracy

In case of a triangular degeneracy of q towards a triangle t,

$$(\exists i \in \{A, B, C, D\}) \quad \lim_{q \to t} \mathcal{A}_i = 0 \tag{6.21}$$

which implies in particular that $\min_i \mathcal{A}_i$ tends to 0, while neither $|q|_2$ nor $h_{\max}$ do; thus, by definition, $\mathcal{Q}$ diverges to $+\infty$. In other words, $\mathcal{Q}$ detects triangular degeneracies, while neither ζ nor ι do. Consequently, $\mathcal{Q}$ is not continuous with any underlying triangle quality measure, unlike ζ and ι.

Remark 6.1. Clearly, this behavior is due to the fact that the denominator of $\mathcal{Q}$ is no longer the entire area of the quadrilateral, but the minimal triangular area. Variations of ζ and ι can be designed consequently, if the goal is to detect triangular degeneracy.

An example of how ζ, ι and $\mathcal{Q}$ behave in case of triangular degeneracy is provided by the following example:

Example 6.1. Let $\Diamond_x$ denote the kite such that ABC is a unitary equilateral triangle, while ACD is iscosceles in D, with $AD = CD = x$, as illustrated Figure 6. Obviously, x must belong to $]\,\frac{1}{2}, +\infty\,[$ but, since the aim is here to

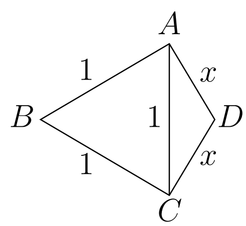

Figure 6: Kite $\Diamond_x$.

examine the case of a triangular degeneracy, the interval is limited to $]\,\frac{1}{2}, 1\,[$. It is straightforward to determine ζ, ι and $\mathcal{Q}$ as function of x when $q = \Diamond_x$. In fact,

$$\min_i \mathcal{A}_i(x) = \frac{1}{4}\sqrt{x^2 - \frac{1}{4}} \tag{6.22}$$

$$p(x) = 1 + x \tag{6.23}$$

$$|q|_2(x) = \sqrt{2(1+x^2)} \tag{6.24}$$

$$|q|_\infty(x) = 1 \tag{6.25}$$

$$\mathcal{A}(x) = \frac{\sqrt{3}}{4} + \frac{1}{4}\sqrt{x^2 - \frac{1}{4}}. \tag{6.26}$$

Hence, when $x \to \frac{1}{2}$, both ζ and ι tend towards finite values (respectively, $3\sqrt{3}$ and $2\sqrt{3}$), while $\mathcal{Q} \to +\infty$.

7 Adaptation of κ_2 to right isosceles reference triangles

An interesting approach to estimate triangle quality has been proposed by various authors (*cf.* [2, 6, 8]), based on the singular values of a matrix which expresses the affine transformation between the mesh element and a given reference element. More precisely, these works have focused on the case where the reference element is a regular simplex, since this element is generally considered to be the best possible for isotropic simplicial meshes. An in-depth examination of the variations of such quality measures, as well as a comparison with other quality measures has been made in [10].
It does not seem that the full derivation, in the case where the reference element is a right isosceles triangle, has yet been done by other authors; one reason is that such elements do not generally correspond to the kind of triangles that are wished in the context of finite element analysis. However, in the goal of extending this measure to quadrilateral meshes, such elements become naturally the desired ones.

7.1 Construction

As described in [2], the *edge-matrix* of a triangle t is defined by:

$$T_0 = \begin{pmatrix} x_F - x_E & x_G - x_E \\ y_F - y_E & y_G - y_E \end{pmatrix} \tag{7.1}$$

and let W be the edge-matrix of a reference isosceles right triangle, for example

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7.2}$$

meaning that W is, simply, the identity matrix of $\mathbb{R}^2$. Hence, $T_0 W^{-1} = T_0$ is the matrix that maps the reference element into t. Using the same ideas as in [2], it is possible to define matrix-norms based on the singular values σ of T_0. Obviously, the symmetry which arises when W is an equilateral triangle vanishes with this new reference element. In particular, t is considered as being optimal only if the right angle is in A. This property allows a strict control, not only over the shape of t, but also on the vertex at which the right angle should be. The singular values are given by the positive square-roots of the eigenvalues of the positive definite matrix $T_0^T T_0$. Now,

$$T_0^T T_0 = \begin{pmatrix} g^2 & \overrightarrow{EF} \cdot \overrightarrow{EG} \\ \overrightarrow{EF} \cdot \overrightarrow{EG} & f^2 \end{pmatrix}, \tag{7.3}$$

where $\cdot$ denotes the usual scalar product. The singular values σ of T_0 are thus obtained from the characteristic equation of $T_0^T T_0$ as

$$\sigma^4 - (f^2 + g^2)\sigma^2 + f^2 g^2 - \left(\overrightarrow{EF} \cdot \overrightarrow{EG}\right)^2 = 0 \tag{7.4}$$

or, equivalently,

$$\sigma^4 - (f^2 + g^2)\sigma^2 + 4{\mathcal{A}_t}^2 = 0. \tag{7.5}$$

Hence,

$$\sigma_1^2 + \sigma_2^2 = f^2 + g^2 \tag{7.6}$$

and $\sigma_1\sigma_2 = 2\mathcal{A}_t$ where σ_1^2 and σ_2^2 $(0 < \sigma_1 \leq \sigma_2)$ are the two roots of (7.5). A quality measure can be constructed from the condition number of any unitarily invariant norm of the matrix T_0 (*cf.* [2]). One such family is derived from the SCHATTEN p-norms defined by:

$$N_p(T_0) = (\sigma_1^p + \sigma_2^p)^{1/p} \quad , \quad p \in [1, +\infty[. \tag{7.7}$$

The case $p = 2$ is the FROBENIUS norm, the limiting case $p \to \infty$ is the spectral norm and the case $p = 1$ is the trace norm. A non-normalized quality measure is given by the condition number $\kappa_p(T_0)$ which is defined as

$$\kappa_p(T_0) = \left[(\sigma_1^p + \sigma_2^p)\left({\sigma_1}^{-p} + {\sigma_2}^{-p}\right)\right]^{1/p}. \tag{7.8}$$

In the particular case $p = 2$, using (1.2), it follows that

$$\kappa_2(T_0) = \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1\sigma_2} = \frac{f^2 + g^2}{2\mathcal{A}_t} = \frac{f^2 + g^2}{fg\sin\eta} \tag{7.9}$$

thus $\kappa_2(T_0)$ depends only on metric and angular parameters of t, and therefore can be denoted unambiguously as a function of t. In order to avoid the confusion with the "classic" κ_2 triangle quality measure, with equilateral reference element, a slighty different notation shall be used, *e.g.*,

$$\kappa_2^{\perp}(t) = \kappa_2(T_0). \tag{7.10}$$

Now, assuming that $\xi = \frac{f}{g}$, which is allowed since t is non degenerate and thus $g \neq 0$, it then follows that

$$\kappa_2^{\perp}(t) = \frac{\xi^2 + 1}{\xi\sin\eta} = \left(\xi + \frac{1}{\xi}\right)\frac{1}{\sin\eta} \tag{7.11}$$

or, in entirely angular terms, since (1.2) shows that $\xi = \frac{\sin\phi}{\sin\psi}$,

$$\kappa_2^{\perp}(t) = \left(\frac{\sin\phi}{\sin\psi} + \frac{\sin\psi}{\sin\phi}\right)\frac{1}{\sin\eta} = \frac{\sin^2\phi + \sin^2\psi}{\sin\eta\sin\phi\sin\psi}. \tag{7.12}$$

Remark 7.1. As demonstrated in [10], the following angular identity arises in the context of equilateral reference elements:

$$\kappa_2(t) = \frac{\sin^2\eta + \sin^2\phi + \sin^2\psi}{\sin\eta\sin\phi\sin\psi} \tag{7.13}$$

and is invariant through angle permutation[9], unlike (7.12). This is intuitively clear, since equilateral triangles are $\mathfrak{S}_3$-invariant, while right isosceles triangles are not. Therefore, since η, ϕ and ψ are supplementary, the surface $z = \kappa_2(t)$ can be equivalently represented as a function of any angle pair of t, while this is no longer true for $z = \kappa_2^{\perp}(t)$, for which at least one angle must be specifically chosen. In particular, if one angle is chosen to be η (the one which is right, ideally, for $\kappa_2^{\sqsubset}$), then the second is arbitrary and can be, *e.g.*, ϕ. In this case, it follows from (7.12) and (7.13), respectively,

$$\kappa_2^{\perp}(t) = \frac{\sin^2\phi + \sin^2(\eta+\phi)}{\sin\eta\sin\phi\sin(\eta+\phi)} \tag{7.14}$$

and

$$\kappa_2(t) = \frac{\sin^2\eta + \sin^2\phi + \sin^2(\eta+\phi)}{\sin\eta\sin\phi\sin(\eta+\phi)}. \tag{7.15}$$

7.2 *Extremum*

According to (7.11), consider the mapping

$$\begin{array}{rccc} k: & \mathbb{R}_+^* \times]\,0,\pi\,[& \longrightarrow & \mathbb{R}_+^* \\ & (\xi,\eta) & \longmapsto & \left(\xi + \frac{1}{\xi}\right)\dfrac{1}{\sin\eta} \end{array} \tag{7.16}$$

which is $\mathcal{C}^\infty$ over the open domain $\mathbb{R}_+^* \times]\,0,\pi\,[$; hence, any local *extremum* of k is attained at a stationary point. The first order derivatives are:

$$\frac{\partial k}{\partial\xi}(\xi,\eta) = \left(1 - \frac{1}{\xi^2}\right)\frac{1}{\sin\eta} \tag{7.17}$$

$$\frac{\partial k}{\partial\eta}(\xi,\eta) = -\left(\xi + \frac{1}{\xi}\right)\frac{\cos\eta}{\sin^2\eta} \tag{7.18}$$

and, given the definition domain, the only stationary point is $(1, \frac{\pi}{2})$. Again, the nature of this point can be examined by the means of the hessian matrix of, assembled with the second order derivatives of k:

$$\frac{\partial^2 k}{\partial\xi^2}(\xi,\eta) = \frac{2}{\xi^3\sin\eta} \tag{7.19}$$

$$\frac{\partial^2 k}{\partial\eta^2}(\xi,\eta) = \left(\xi + \frac{1}{\xi}\right)\frac{\sin^2\eta + 2\cos^2\eta}{\sin^3\eta} \tag{7.20}$$

$$\frac{\partial^2 k}{\partial\xi\partial\eta}(\xi,\eta) = -\left(1 - \frac{1}{\xi^2}\right)\frac{\cos\eta}{\sin^2\eta} \tag{7.21}$$

which gives, when $(\xi,\eta) = (1, \frac{\pi}{2})$,

$$\frac{\partial^2 k}{\partial\xi^2}\left(1, \frac{\pi}{2}\right) = \frac{\partial^2 k}{\partial\eta^2}\left(1, \frac{\pi}{2}\right) = 2 \tag{7.22}$$

[9] or, in less rigorous but simpler terms, symmetrical.

$$\frac{\partial^2 k}{\partial \xi \partial \eta}\left(1, \frac{\pi}{2}\right) = 0. \tag{7.23}$$

Thus, the hessian determinant is equal to $4 > 0$ and the first diagonal entry is $2 > 0$. Hence, the hessian matrix is locally positive definite around the critical point, which therefore corresponds to a strict local *minimum* of k. Since k is $\mathcal{C}^\infty$ over its open and connected definition domain, the unicity of the critical point ensures that this *minimum* is, also, absolute. In other words, $\kappa_2^\perp(t)$ is minimal only for right ($\eta = \frac{\pi}{2}$) isosceles ($\xi = 1 \Leftrightarrow f = g$) triangles. In this case, the value of $\kappa_2^\perp(t)$ is, obviously 2, which provides the normalization coefficient.

Remark 7.2. If t is equilateral, then $\kappa_2^\perp(t) = \frac{4}{\sqrt{3}} \gtrapprox 2.31 > 2$, as expected since the reference element is no longer equilateral.

7.3 Asymptotic behavior

A needle degeneracy occurs when one, and only one, of the angles of t tends to 0 (*cf.* [10]). This implies none of them tends to π, otherwise there would be two angles tending to 0. Hence, one and only one of the sines in (7.12) tends to 0, implying that the numerator tends to a non-zero value, while the denominator tends to 0. Therefore, $\kappa_2^\perp(t) \to +\infty$.

In the case where t flattens, *i.e.*, one of its angles tends to π, either $\eta \to \pi$ (flattens in E), or $\eta \to 0$ (flattens in F or G). In both cases, $\sin\eta \to 0$ thus, combined with the fact that $\xi + \frac{1}{\xi} \geq 2$, it follows from (7.9) that $\kappa_2^\perp(t) \to +\infty$.

Remark 7.3. In the case where t is a right triangle, but not necessarily right isosceles in E, it follows from (7.11) that:

- if t is right in E, then $\kappa_2^\perp(t) = \frac{f^2+g^2}{fg}$, which equals 2 if and only if $f = g$, and tends to $+\infty$ as either $\frac{f}{g}$ or $\frac{f}{g}$ does (needle degeneracy);

- if t is right in F, then $\kappa_2^\perp(t) = \frac{e^2+2g^2}{eg}$, which at best equals $2\sqrt{2}$, when $e = g\sqrt{2}$, and tends to $+\infty$ as either $\frac{e}{g}$ or $\frac{g}{e}$ does (needle degeneracy). In particular, if t is also isosceles, then $\kappa_2^\perp(t) = 3$. For symmetry reasons, the results are obviously the same if t is right in G.

Figure 7 provides a graphic comparison between the respective behaviors of κ_2 (*cf.* [10] for details) and $\kappa_2^\perp$. In fact, three different $z = \kappa_2^\perp$ surfaces can be defined, depending on which couple of angular variables is picked, because of the asymmetry explained in Remark 7.1. Therefore, at least concerning $\kappa_2^\perp$, either an η-axis must be specified, or neither of the two axes[10] actually are associated to η. In Figure 7, η is in abscissa and, as expected, the two surfaces have different bottoms: $(\frac{\pi}{2}, \frac{\pi}{4}, 1)$ for $z = \kappa_2^\perp$ and $(\frac{\pi}{3}, \frac{\pi}{3}, 1)$ for $z = \kappa_2$.

[10]in fact, in this case, η is implicitly represented along the $x + y = \pi$ axis.

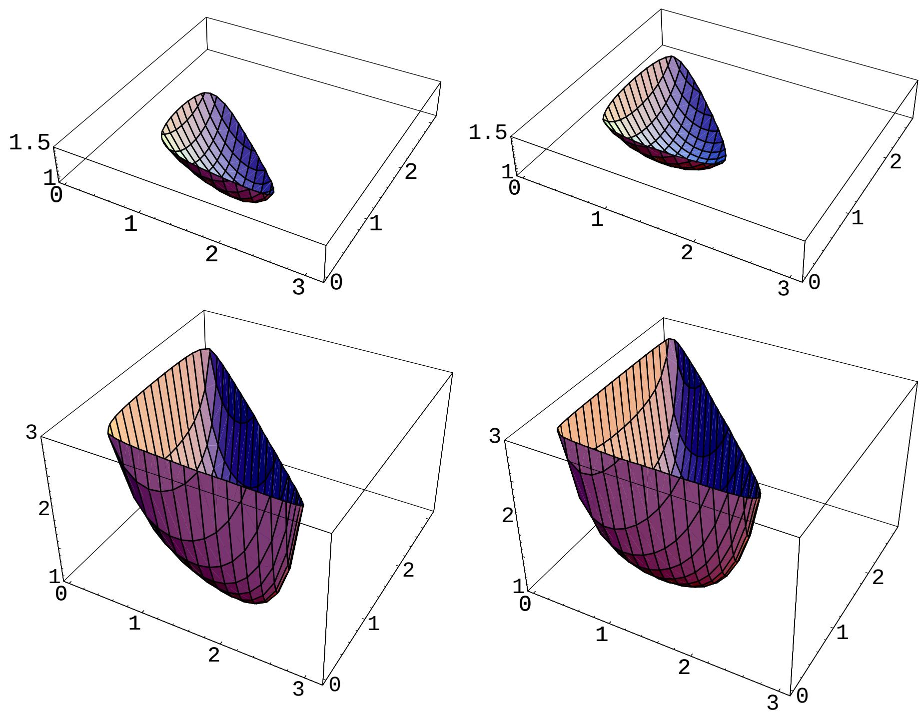

Figure 7: Surfaces $z = \frac{\kappa_2^\perp}{2}$ (left) and $z = \frac{\kappa_2}{2}$ (right) as functions of η (abscissa) and either ϕ or ψ (ordinate), for $z < 1.5$ (up) and $z < 3$ (down).

8 Extending κ_2 to quadrilaterals

Section 7 provides a matrix-based triangle quality measure to the case where the reference element is a right isosceles triangle with, in addition, a specific control over which edge is the hypotenuse. The main motivation of this modification is to allow, in a second step, the adaptation of κ_2 to quadrilaterals.

8.1 Construction

Considering the generic planar quadrilateral q, four different triangles might be evaluated by the means of $\kappa_2^{\perp}$: $t_A = DAB$, $t_B = ABC$, $t_C = BCD$ and $t_D = CDA$, with respective edge-matrices T_A, T_B, T_C and T_D. Now, it follows from (7.1) that

$$T_A + T_B + T_C + T_D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{8.1}$$

i.e.,

$$T_3 = -T_0 - T_1 - T_2. \tag{8.2}$$

In other words, it is unnecessary to evaluate the four edge-matrices at each vertex of the quadrilateral, since any of them is a linear combination of the three other ones. This simply means that, given three vertex angles and edge ratios, the quadrilateral is fully determined, up to homothecy.
Now, considering $\kappa_2^{\perp}(t_i)$, as it has been previously modified for right isosceles triangles, the qualities of each of these four triangles are, respectively,

$$\kappa_2^{\perp}(t_A) = \frac{a^2+d^2}{ad\sin\alpha}, \quad \kappa_2^{\perp}(t_B) = \frac{a^2+b^2}{ab\sin\beta}, \tag{8.3}$$

$$\kappa_2^{\perp}(t_C) = \frac{b^2+c^2}{bc\sin\gamma}, \quad \kappa_2^{\perp}(t_D) = \frac{c^2+d^2}{cd\sin\delta}. \tag{8.4}$$

According to (8.1), $\kappa_2^{\perp}(t_D)$ is related to $\kappa_2^{\perp}(t_A)$, $\kappa_2^{\perp}(t_B)$ and $\kappa_2^{\perp}(t_C)$, but this dependency is no longer linear, since singular values and, hence, polynomial equations, are involved. Therefore, although it might appear as more elegant to design a quadrilateral quality measure, depending only on three of the underlying triangle qualities, it is certainly much more costly. For this reason, a more realistic and certainly more efficient idea is to take into account the four qualities. For instance, one might take the worst of these qualities, in other words their max norm:

$$\kappa_2^{\infty}(q) = \max\left(\kappa_2^{\perp}(t_A), \kappa_2^{\perp}(t_B), \kappa_2^{\perp}(t_C), \kappa_2^{\perp}(t_D)\right). \tag{8.5}$$

Remark 8.1. As always, the problem with the max norm is that, provided the worst triangle remains the same, it is intrisically unable to take into account quality variations of the three other ones, as illustrated by Figure 8: κ_2^{∞} cannot detect the fact than a quadrilateral is "less" distorted than the other.

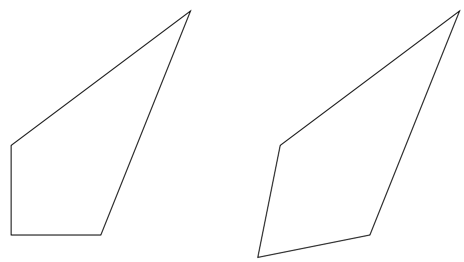

Figure 8: Both quadrilaterals share the same κ_2^∞.

A natural attempt to address the case mentioned in Remark 8.1 is to consider the arithmetic mean instead of the max norm:

$$\kappa_2^1(q) = \frac{\kappa_2^\perp(t_A) + \kappa_2^\perp(t_B) + \kappa_2^\perp(t_C) + \kappa_2^\perp(t_D)}{4}. \tag{8.6}$$

Remark 8.2. The choice of the arithmetic mean is arbitrary without any further justification. One might, *e.g.*, prefer to use the euclidean norm instead.

8.2 *Extremum*

It has been proved in Section 7 that

$$(\forall\, i \in \{A, B, C, D\}) \quad \kappa_2^\perp(t_i) \geq 2 \tag{8.7}$$

with equality if and only if t_i is a right isosceles triangle. Hence, by definition,

$$\kappa_2^1(q) \geq \frac{2+2+2+2}{4} = 2 \tag{8.8}$$

with equality if and only if all $\kappa_2^\perp(t_i)$ are equal to 2. In addition,

$$\kappa_2^\infty(q) = 2 \iff (\forall\, i \in \{A, B, C, D\}) \quad \kappa_2^\perp(t_i) = 2 \tag{8.9}$$

whence both κ_2^∞ and κ_2^1 reach their unique absolute *minimum* only when q is a square. Moreover, they have the same *minimum*, 2, which provides the normalization coefficient.

8.3 Asymptotic behavior

8.3.1 Parallelogram

As explained in Paragraph 5.2, when q is parallelogram it follows that $\sin\alpha = \sin\beta = \sin\theta$; in addition, $a = c$ and $b = d$, whence

$$(\forall\, i \in \{A, B, C, D\}) \quad \kappa_2^\perp(t_i) = \frac{a^2+b^2}{ab\sin\theta} \tag{8.10}$$

from which it follows that

$$\kappa_2^\infty(q) = \kappa_2^1(q) = \frac{a^2+b^2}{ab\sin\theta} = \frac{|q|_2^2}{2\mathcal{A}} \tag{8.11}$$

or, equivalently,

$$\kappa_2^\infty(q) = \kappa_2^1(q) = \frac{|q|_0^2+|q|_\infty^2}{|q|_0|q|_\infty\sin\theta} \tag{8.12}$$

from which the following expression in terms of stretching and torsion of κ_2^1 for parallelograms arises:

$$\kappa_2^\infty(q) = \kappa_2^1(q) = \frac{1}{\sin\theta}\left(\tau+\frac{1}{\tau}\right). \tag{8.13}$$

Remark 8.3. In the context of the κ_2 triangle quality measure, with an equilateral reference element, it is shown in [10] that the following identity holds:

$$\kappa_2(t) = \frac{|t|_2^2}{2\sqrt{3}\mathcal{A}_t}, \tag{8.14}$$

which appears to be similar to (8.11), up to a constant factor. It is quite satisfactory to obtain the same result, up to a constant factor, for triangles[11] and parallelograms.

8.3.2 Rectangle

If q is a rectangle, then (8.13) can obviously be simplified as follows:

$$\kappa_2^\infty(q) = \kappa_2^1(q) = \tau+\frac{1}{\tau} \tag{8.15}$$

thus κ_2^∞ and κ_2^1 reduce to the same function, denoted as $\kappa_{2\Box}$, which depends only on the stretching factor of q, and is obviously strictly increasing over $[1,+\infty[$. In addition, its asymptotic behavior is given by

$$\kappa_{2\Box}(\tau) \underset{\tau\to+\infty}{\sim} \tau \underset{\tau\to+\infty}{\sim} \zeta_\Box(\tau), \tag{8.16}$$

which implies, in particular, that $\kappa_{2\Box}$ diverges towards $+\infty$ when τ tends to $+\infty$.

8.3.3 Rhombus

It is also interesting to notice that, in the case where q is a rhombus, (8.13) show that both κ_2^∞ and κ_2^1 reduce to a function of θ:

$$\kappa_2^\infty(q) = \kappa_2^1(q) = \kappa_{2\diamond}(\theta) = \frac{2}{\sin\theta} \tag{8.17}$$

[11]in the usual case, when the reference element is equilateral.

which strictly decreases (resp. decreases) on $]0, \frac{\pi}{2}]$ (resp. $[\frac{\pi}{2}, \pi[$) and is equivalent to $\frac{2}{\theta}$ (resp. $\frac{2}{\pi-\theta}$) when $\theta \to 0^+$ (resp. $\theta \to \pi^-$). In particular, $\kappa_{2\diamond}$ diverges to $+\infty$ when θ tends to either 0^+ or π^-.

The relative sensitivities of κ_2^∞ and κ_2^1 to stretching and torsion are now discussed, by solving $\kappa_{2\Box}(\tau) = \kappa_{2\diamond}(\theta)$, *i.e*,

$$\tau + \frac{1}{\tau} = \frac{2}{\sin\theta} \quad \Longleftrightarrow \quad \frac{\tau^2 \sin\theta - 2\tau + \sin\theta}{\tau \sin\theta} = 0 \tag{8.18}$$

and solving for τ reduces to finding the roots of the polynomial $X^2 \sin\theta - 2X + \sin\theta$. Obviously, these roots are $\frac{1 \pm |\cos\theta|}{\sin\theta}$, corresponding to a double root only when $\theta = \frac{\pi}{2}$ and, hence, $\tau = 1$, *i.e.*, if and only if q is a square. Conversely, $\tau = 1$ implies $\sin\theta = 1$ thus $\theta = \frac{\pi}{2}$. In the case where $\theta \neq \frac{\pi}{2}$, it is clear that $1 + |\cos\theta| > \sin\theta > 0$, hence $\frac{1+|\cos\theta|}{\sin\theta}$ complies with the fact that $\tau > 1$. Now, $\theta \in]0, \pi[\setminus \{\frac{\pi}{2}\}$ implies $0 < \sin\theta < 1$ thus $\sin^2\theta < \sin\theta$ and

$$1 - |\cos\theta| < 1 - \cos^2\theta = \sin^2\theta < \sin\theta. \tag{8.19}$$

Therefore, $\frac{1+|\cos\theta|}{\sin\theta} < 1$, hence for any $\theta \in]0, \pi[$, (8.18) has the following unique solution:

$$\tau = \frac{1 + |\cos\theta|}{\sin\theta}. \tag{8.20}$$

Moreover, solving (8.18) for θ obviously brings two possible solutions, $\arcsin \frac{2\tau}{\tau^2+1}$ and $\pi - \arcsin \frac{2\tau}{\tau^2+1}$. In fact, they both correspond to the same shape for q, since they are supplementary.

Figure 9 represents the respective stretching factor *vs.* torsion sensitivities of ζ and κ_2^1, using (5.40) and (8.20). Both graphs have a salient point in $(\frac{\pi}{2}, 1)$, since neither the absolute value (for ζ) nor the square root (for κ_2^1) functions are $\mathcal{C}^1$ in 0.

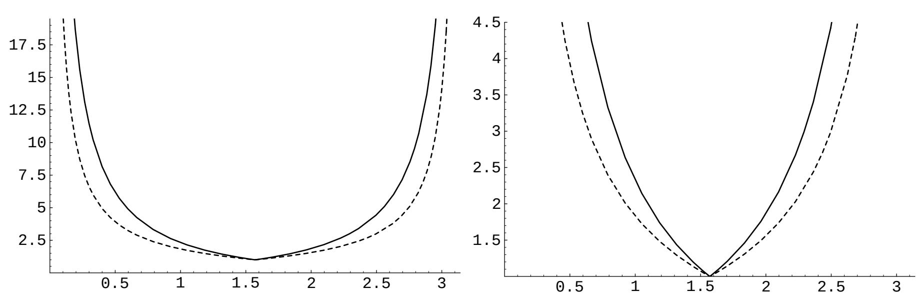

Figure 9: Graphs at two different scales providing the respective sensitivities of ζ (dotted) and κ_2^1 (solid) to stretching *vs.* torsion.

8.3.4 Triangular degeneracy

In case of a triangular degeneracy of q, *i.e.*, at least one of its angles tends to π, it follows immediately from (8.3) and (8.4) that at least one of the $\kappa_2^{\perp}(t_i)$ tends to $+\infty$, as do $\kappa_2^\infty(q)$ and $\kappa_2^1(q)$. In other words, both κ_2^∞ and κ_2^1 diverge in case of triangular degeneracy.

Remark 8.4. An other quality measure could be defined by generalizing (8.11) to all quadrilaterals. In the case of a triangular degeneracy, this measure would not diverge because the denominator (the area) would not tend to 0, while the numerator (the sum of all squared edge lengths) remains bounded. Moreover, it would not be continuous with the κ_2 triangle quality measure, even after adjusting normalization coefficients, since $\mathcal{A} \to \mathcal{A}_t$ while $|q|_2^2 \not\to |t|_2^2$. This makes a clear difference with ζ, for which triangular continuity is ensured by the fact that $p \to p_t$.

9 Numerical results

Because of the four-dimensionality of Q_1, it is impossible to provide comprehensive plottings of quadrangle quality measures, *e.g.* as in [10]. The alternate approach used here is to test these measures over a representative set of quadrilaterals. In addition, τ, σ, ζ, ι, $\mathcal{Q}$, κ_2^1 and κ_2^∞ are tested on various quadrilateral meshes.

9.1 Reference quadrilaterals

The proposed set of reference elements consists in the previously introduced quadrilaterals $\square$, $\circledcirc$, $\Diamond$ and $\Diamond_x$ (with $x = 0.51$ and $x = 0.501$), along with:

- $\boxed{2}$ and $\boxed{100}$, rectangles with respective stretching factors of 2 and 100;
- q_1 and q_2, denoting the two quadrilaterals of Figure 8;
- $\Diamond$, a rhombus such as $AB = 5AC$.

The corresponding normalized quality measures are given Table 1, from which several remarks can be made:

– as mentioned before, neither τ nor σ make sense as quality measures in a general context; however, when considered together, they provide valuable information about quadrilateral shape along two “dimensions” of Q_1;

– as expected, neither ζ nor ι detect the triangular degeneracies which appear in $\Diamond_{0.51}$ and, more markedly, in $\Diamond_{0.501}$. The fact that, nevertheless, ι grants worse qualities to these quadrangles than ζ does is due to the “unsmoothing effect” implied by the presence of $|q|_\infty$ in the former;

– as expected, κ_2^1 and κ_2^∞ are disconnected as soon as asymmetry appears. In this case, κ_2^∞ is more sensitive to distortion than κ_2^1. Moreover, although κ_2^1 is supposed to be more discriminant than κ_2^∞ when the worst part[12] of q is invariant, this difference is not significant between q_1 and q_2;

[12] in the sense of the κ_2 triangle quality measure.

	τ	σ	$\frac{1}{4}\zeta$	$\frac{1}{2}\iota$	$\frac{1}{4\sqrt{2}}\mathcal{Q}$	$\frac{1}{2}\kappa_2^1$	$\frac{1}{2}\kappa_2^\infty$
$\square$	1	1	1	1	1	1	1
$\boxed{2}$	2	1	1.125	1.5	1.25	1.25	1.25
$\boxed{100}$	100	1	25.5	50.5	50	50.01	50.01
$\Diamond_1$	1	1.5	1.155	1.155	1.414	1.155	1.155
$\Diamond_5$	1	7.841	5.025	5.025	7.071	5.025	5.025
$\Diamond_{0.51}$	1.961	1.261	1.18	1.562	5.585	1.588	2.588
$\Diamond_{0.501}$	1.996	1.415	1.255	1.672	17.68	2.969	7.933
$\textcircled{q}$	7.4	1	1.885	3.389	33.9	2.957	3.604
q_1	2.693	1.484	1.437	2.152	4.456	1.598	1.929
q_2	2.112	1.824	1.498	2.085	3.428	1.602	1.929

Table 1: Qualities of a representative set of quadrilaterals.

– $\mathcal{Q}$ is extremely sensible, which is not a problem *per se*, but in this case one can wonder whether it is justified to obtain, roughly, $\mathcal{Q}(\textcircled{q}) \approx 10\mathcal{Q}(q_2)$. In addition, $\mathcal{Q}(q_1)$ is significantly worse than $\mathcal{Q}(q_2)$ as opposed to, *e.g.*, the results obtained with κ_2-based quality measures. Depending on application, one might decide which of q_1 or q_2 should be considered as best, but $\mathcal{Q}$ seems to be slightly unbalanced overall

9.2 Quadrilateral meshes

An other perspective on mesh quality measures is to examine how they behave on actual meshes. In this goal, it will be made use of six different quadrangular meshes, denoted as L_1 to L_6, of a given planar domain, as illustrated by Figure 10. These meshes differ in regularity, density an isotropy, and these variations should appear in terms of mesh quality. There are numerous ways to evaluate the quality of an entire mesh from that of its elements; discussing them is beyond the scope of this article. The most natural approach consists in considering the range[13] and the arithmetic mean of element quality measures across the entire mesh. The corresponding results are given in Table 2. From these numerous results, it appears that:

– clearly, $\mathcal{Q}$ and κ_2^∞ are much more discriminant than the other quality estimates. For example, both L_1 and L_4 contain very distorted elements (as confirmed by worst torsion and skewness) and the respective *maxima* of $\mathcal{Q}$ and κ_2^∞ across these meshes are much greater than for the other quality

[13] *i.e.*, the best and the worst element qualities.

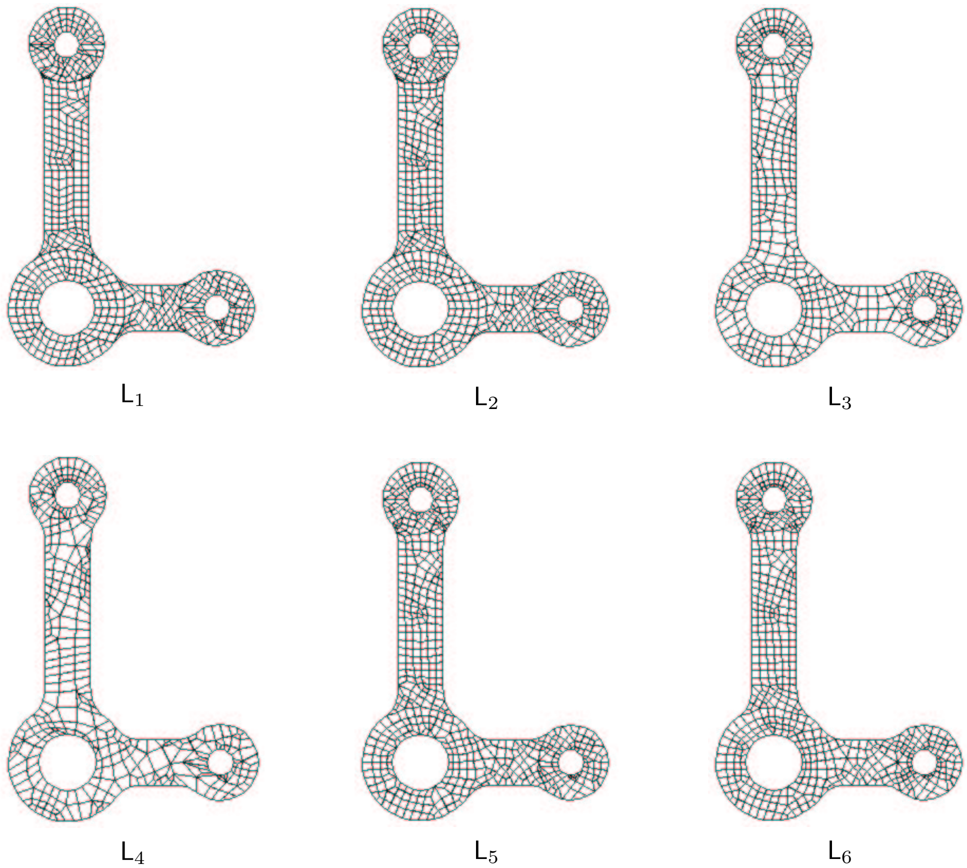

Figure 10: Six different quadrilateral meshes of the same domain. *Data courtesy of* INRIA Rocquencourt, France.

	$\overline{\tau}$ $-$ $+$	$\overline{\sigma}$ $-$ $+$	$\frac{1}{4}\overline{\zeta}$ $-$ $+$	$\frac{1}{2}\overline{\iota}$ $-$ $+$	$\frac{1}{4\sqrt{2}}\overline{\mathcal{Q}}$ $-$ $+$	$\frac{1}{2}\overline{\kappa_2^1}$ $-$ $+$	$\frac{1}{2}\overline{\kappa_2^\infty}$ $-$ $+$
L_1	1.66 1 5.81	1.31 1 12.1	1.2 1 10.3	1.5 1 15.5	2.51 1 103	1.35 1 27.2	2.64 1 162
L_2	1.65 1.02 6.23	1.2 1 5.38	1.12 1 4.78	1.39 1.01 7.82	1.93 1.02 18.8	1.2 1 5.7	2.02 1 24.1
L_3	1.63 1.04 3.48	1.13 1 2.13	1.07 1 1.6	1.33 1.02 2.34	1.66 1.03 7.05	1.12 1 2.14	1.8 1 4.48
L_4	1.94 1.02 6.39	1.19 1 2.65	1.15 1 3.47	1.52 1.03 6.21	2.22 1.06 36	1.26 1 5.51	2.24 1 16.7
L_5	1.56 1.03 4.43	1.14 1 2.1	1.07 1 2.04	1.3 1.02 3.57	1.66 1.03 7.08	1.12 1 2.44	1.78 1 6.29
L_6	1.51 1.02 3.7	1.1 1 2.07	1.05 1 1.49	1.26 1.01 2.35	1.59 1.02 7.12	1.09 1 2.13	1.69 1 4.48

Table 2: Arithmetic mean and range of quality measures across six different quadrilateral meshes of the same domain.

measures. In fact, these specific examples exhibit a strong correlation between $\mathcal{Q}$ and κ_2^∞ which is not a general rule, as exhibited by Table 1;

- the unbalanced behavior of $\mathcal{Q}$, suspected from the results of Table 1, is not confirmed here;
- on the contrary, κ_2^1 and κ_2^∞ are largely disconnected: the trend observed in Table 1 for some configurations is confirmed across entire meshes;
- L_3 is considered as very good mesh by ζ but not by $\mathcal{Q}$ nor κ_2^∞. A close visual examination of this mesh shows that only triangular degeneracies can be observed, explaining why ζ does not detect them.

Conclusions and future work

The results demonstrated in this article concerning quadrilateral quality measures are summarized in Table 3. Column "$= 1$" indicates which particular element optimizes the normalized quality; columns "$\boxed{\infty}$ "and $\lozenge$ provides the asymptotic behavior of the normalized quality when, respectively, the element is a rectangle with stretching factor τ tending to $+\infty$ and a rhombus with torsion θ tending to 0; column $\lozenge$; column "$\rightarrow \triangle$" indicates whether measure diverges or not in the case of a triangular degeneracy of the quadrilateral. Depending on the specific needs of the user, Table 3 allows one to choose which quality measure fits one's specific needs. In particular whether divergence to $+\infty$ in the case

	$=1$	$\boxed{\infty}$	$\Diamond_0$	$\rightarrow \triangle$
τ	rhombus	$\sim \tau$	$=1$	no
σ	inscribed	$=1$	$= \frac{\pi}{2\theta}$	no
$\frac{1}{4}\zeta$	square	$\sim \frac{\tau}{4}$	$\sim \frac{1}{\theta}$	no
$\frac{1}{2}\iota$	square	$\sim \frac{\tau}{2}$	$\sim \frac{1}{\theta}$	no
$\frac{1}{4\sqrt{2}}\mathcal{Q}$	square	$\sim \frac{\tau}{2}$	$\sim \frac{\sqrt{2}}{\theta}$	yes
$\frac{1}{2}\kappa_2^1$	square	$\sim \frac{\tau}{2}$	$\sim \frac{1}{\theta}$	yes
$\frac{1}{2}\kappa_2^\infty$	square	$\sim \frac{\tau}{2}$	$\sim \frac{1}{\theta}$	yes

Table 3: Summary of quadrilateral quality measures.

of a triangular degeneracy is desireable or not. Moreover, it seems that, in the case of $\mathcal{Q}$, considering $@$ as five times worse than $\Diamond_{0.51}$ is certainly subject to caution. Overall, κ_2^1 and κ_2^∞ appear to be the best-behaved planar quadrilateral quality measures, in the case where triangular degeneracy detection is desireable.

Among other interesting questions remaining to be addressed are:

- point-placement strategies; in other words, given three vertices and a quality measure, what is (are) the optimal fourth vertex position(s) ?
- possible extensions to quadrilateral-sided hexahedrons;
- the correlation of these results to actual numerical quadrilateral-based numerical finite element or finite volume computations.

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